Perpetuities and Annuities: Derivation of shortcut formulas

Outline

Perpetuity formula

A **perpetuity** is a stream of <u>equal cash flows</u> that occur at regular intervals and <u>last for ever</u>

The mathematical derivation of the PV formula

The present value of a perpetuity P with payment C and interest r is given by:

$$PV(P) = \frac{C}{(1+r)^1} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots$$

$$PV(P) = C * \left(\frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \cdots \right)$$

$$PV(P) = C * \sum_{n=1}^{\infty} \frac{1}{(1+r)^n}$$

You may recognize this, from Calculus classes, as a **geometric progression**:

$$X = \sum_{n=1}^{\infty} Z^t$$

Where Z is a positive constant that is less than 1, and X is the sum of the geometric progression

Recall that the sum of such a series actually has a closed-form solution:

$$X = \sum_{n=1}^{\infty} Z^t = \frac{Z}{1-z}$$

The Present Value of the perpetuity can then be written as a geometric progression, where $Z = \frac{1}{1+r}$:

$$PV(P) = C * \sum_{n=1}^{\infty} \frac{1}{(1+r)^n} = C * \frac{Z}{1-z} = C * \frac{\left(\frac{1}{1+r}\right)}{\left(1 - \left(\frac{1}{1+r}\right)\right)} = C * \frac{1}{r}$$

$$\boldsymbol{PV(Perpetuity) = \frac{C}{r}}$$

Derivation of the perpetuity formula using the Law of One Price

To derive the shortcut, we calculate the value of a perpetuity by creating our own perpetuity.

Suppose you could invest $100 in a bank account paying 5% interest per year forever. Suppose also you withdraw the interest and reinvest the $100 every year. By doing this, you can create a perpetuity paying $5 per year.
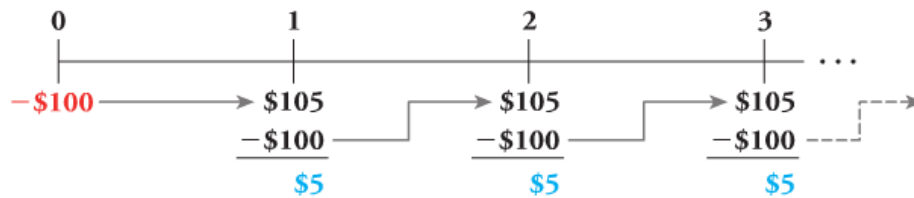

The Law of One Price: the value of the perpetuity must be the same as the cost we incurred to create the perpetuity.

Let's generalize: suppose we invest an amount P in the bank. Every year we can withdraw the interest, $C=r*P$, leaving the principal P. The present value of receiving C in perpetuity is then the upfront cost: P=C/r.

$$PV(Perpetuity) = \frac{C}{r}$$

Annuity formula

An ordinary annuity is a stream of N equal cash flows paid at regular intervals.
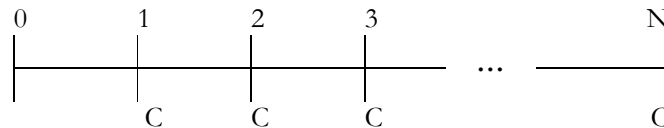
The mathematical derivation of the PV formula

The present value of an N-period annuity A with payment C and interest r is given by:

$$PV(A) = \frac{C}{(1+r)^1} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots + \frac{C}{(1+r)^N}$$

$$PV(A) = C * \sum_{n=1}^{N} \frac{1}{(1+r)^n}$$

You may recognize this, from Calculus classes, as a **finite geometric series**. The formula for the sum of such a series is:

$$X = \sum_{n=1}^{N} Z^t = \frac{Z * (1 - Z^N)}{(1 - z)}$$

The Present Value of the N-period annuity can then be written as a geometric progression, where $Z = \frac{1}{1+r}$:

$$PV(A) = C * \sum_{n=1}^{N} \frac{1}{(1+r)^n} = C * \frac{Z * (1 - Z^N)}{(1 - z)} = C * \frac{\frac{1}{1+r} * \left(1 - \left(\frac{1}{1+r}\right)^N\right)}{\left(1 - \frac{1}{1+r}\right)}$$

This equation can be simplified by multiplying it by $\frac{1+r}{1+r}$ which is to multiply it by 1. Notice that (1+r) is canceled out throughout the equation by doing this. The formula is now reduced to:

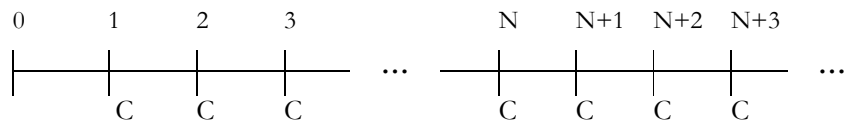$$PV(A) = C * \frac{\left(1 - \left(\frac{1}{1+r}\right)^N\right)}{1 + r - 1}$$

$$\boldsymbol{PV(\textbf{\textit{N-period Annuity}})} = \frac{\boldsymbol{C}}{\boldsymbol{r}} * \left[\boldsymbol{1} - \frac{\boldsymbol{1}}{(\boldsymbol{1+r})^N}\right]$$
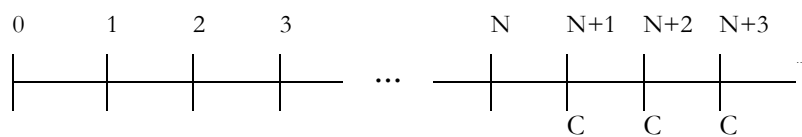
Alternative derivation:

Now consider the time lines for a perpetuity that <u>starts at time 1</u>,
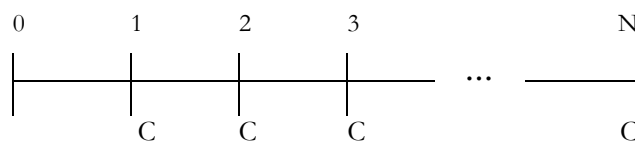
Perpetuity C, t₁

and a perpetuity that <u>starts at time N+1</u>:

Perpetuity C, t_{N+1}

Notice that if we subtract the second time line from the first, we get the time line for an ordinary annuity with N payments:

The present value of an ordinary annuity is then equal to the present value of the first time line minus the present value of the second time line.

The present value of the **Perpetuity C, t₁**, is given by:

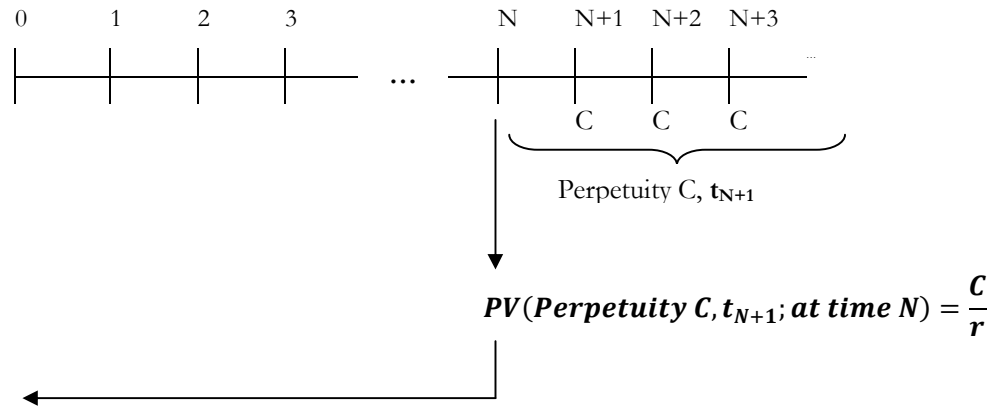$$PV(Perpetuity\ C, t_1) = \frac{C}{r} \quad (1)$$

The present value of the **Perpetuity C, t_{N+1}**, is given by:

$$PV(Perpetuity\ C, t_{N+1}) = \frac{C}{r} * \frac{1}{(1+r)^N} \quad (2)$$

Proof:

If we just look at the time line from N on, it is an ordinary perpetuity that starts at time N + 1. To calculate the present value of the **Perpetuity C, t_{N+1},** we just discount this PV(Perpetuity C,t_{N+1}; at time N) back to time 0:

$$PV(Perpetuity\ C, t_{N+1}; at\ time\ N) = \frac{C}{r}$$

$$PV(Perpetuity\ C, t_{N+1}; at\ time\ 0) = \frac{C}{r} * \frac{1}{(1+r)^N}$$

Subtracting Equation (2) from (1) gives the present value of an ordinary annuity:

$$PV(Nperiod\ Annuity) = \frac{C}{r} * \frac{1}{(1+r)^N} - \frac{C}{r}$$

$$PV(Nperiod\ Annuity) = \frac{C}{r} * \left(1 - \frac{1}{(1+r)^N}\right)$$

Derivation of the annuity formula using the Law of One Price

To derive the shortcut, we calculate the value of an annuity by creating our own annuity.

Suppose you could invest $100 in a bank account paying 5% interest per year. Suppose also you withdraw the interest and reinvest the $100 every year. You might decide after 20 years to close the account and withdraw the principal. By doing this, you can create a 20-year annuity of $5 per year. In addition, you will receive an extra $100 at the end of 20 years.
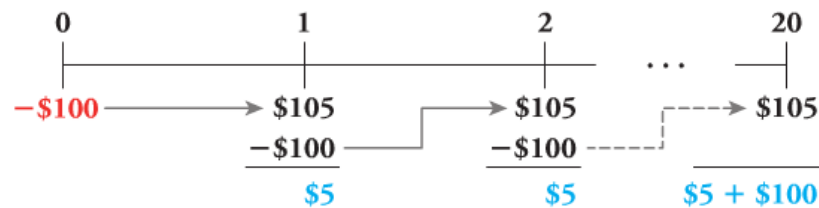
The Law of One Price:

$$Value \ (Asset) \ = \sum PV \ (future \ cash \ flows \ paid \ by \ the \ asset)$$

The present value of all cash flows in our example is $100 (the initial investment required to create these cash flows).

$$\$100 = PV(20\_\text{year annuity of \$5 per year}) + PV(\$100€ \text{ in 20 years})$$

$$PV(20\_\text{year annuity of \$5 per year}) = \$100 - PV(\$100€ \text{ in 20 years})$$

$$= \$100 - \frac{\$100}{(1 + 5\%)^{20}} = \$62.3$$

So the PV of the 20-year annuity of $5 per year is $62.3.

Let's generalize: suppose we invest an amount P in the bank. Every period we can withdraw the interest, $C=r*P$, leaving the principal P. After N periods, we close the account and we get back the original investment P. According to law of one price, P is the present value of all future cash flows.

$$P = PV(\text{Annuity of C for N periods}) + PV(\text{P in period N})$$

$$PV(\text{Annuity of C for N periods}) = P - PV(\text{P in period N})$$

$$= P - \frac{P}{(1 + r)^N}$$

$$= P * \left[1 - \frac{1}{(1 + r)^N}\right]$$

Recall that the periodic payment C is the interest earned every period: $C = r * P$
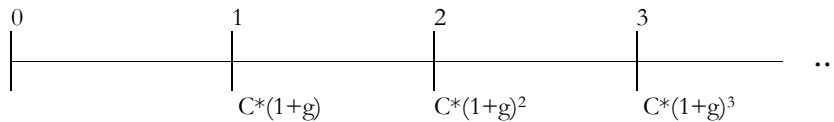
Equivalently,

$$P = \frac{C}{r}$$

The present value of an ordinary Annuity

$$PV(Nperiods\ Annuity) = \frac{C}{r} * \left[1 - \frac{1}{(1+r)^N}\right]$$

Growing Perpetuity formula

A **growing perpetuity** is a stream of cash flows that occur at regular intervals and <u>grow at a constant rate for ever.</u>

The mathematical derivation of the PV formula

The present value of a Growing Perpetuity P with payment C and interest r is given by:

$$PV(GP) = \frac{C}{(1+r)^1} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots$$

$$PV(GP) = \frac{C}{(1+r)} + \sum_{n=2}^{\infty} \frac{C(1+g)^{n-1}}{(1+r)^n}$$

$$PV(GP) = \frac{C}{(1+r)} + \frac{C}{(1+r)} * \sum_{n=1}^{\infty} \frac{(1+g)^n}{(1+r)^n}$$

You may recognize this, from Calculus classes, as a **geometric progression**:

$$X = \sum_{n=1}^{\infty} Z^t$$

Where Z is a positive constant that is less than 1, and X is the sum of the geometric progression

Recall that the sum of such a series actually has a closed-form solution:

$$X = \sum_{n=1}^{\infty} Z^t = \frac{Z}{1-z}$$

The Present Value of the perpetuity can then be written as a geometric progression,

where $Z = \frac{1+g}{1+r}$ and g < r

$$PV(GP) = \frac{C}{(1+r)} + \frac{C}{(1+r)} * \sum_{n=1}^{\infty} \frac{1}{(1+r)^n} = \frac{C}{(1+r)} + \frac{C}{(1+r)} * \frac{\left(\frac{1+g}{1+r}\right)}{\left(1 - \left(\frac{1+g}{1+r}\right)\right)}$$

$$PV(GP) = \frac{C}{(1+r)} + \frac{C}{(1+r)} * \frac{(1+g)}{(r-g)}$$

$$PV(GP) = \frac{C}{(1+r)} * \left(1 + \frac{(1+g)}{(r-g)}\right)$$

$$PV(GP) = \frac{C}{(1+r)} * \left(\frac{(1+r)}{(r-g)}\right)$$

$$PV(Growing\ Perpetuity) = \frac{C}{r-g} \quad ; \quad with\ g < r$$

Derivation of the perpetuity formula using the Law of One Price

To derive the shortcut, we calculate the value of a growing perpetuity by creating our own perpetuity.

Suppose you want to create a perpetuity growing at 2%. You could invest $100 in a bank account paying 5% interest per year forever. At the end of the first year, suppose you withdraw only $3 from the interest ($5). You will have $102to reinvest (2% more than the initial investment). By doing this, you can create a growing perpetuity that starts at $3 and grows 2% per year.
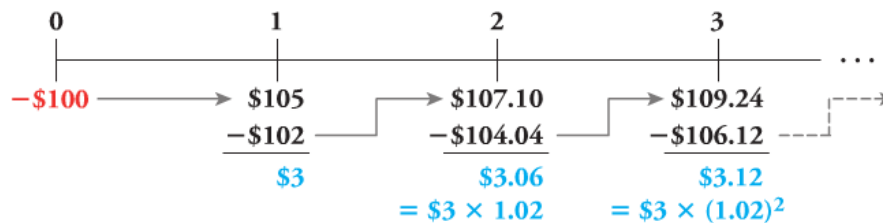
The Law of One Price: the value of the growing perpetuity must be the same as the cost we incurred to create the perpetuity.

Let's generalize: suppose we invest an amount P in the bank. If we want to increase the amout we withdraw each year by g, then P will have to grow by the same factor g.
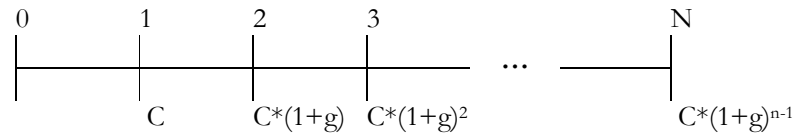
Every year we should reinvest $P(1 + g)$ and withdraw $C = (rP - gP) = P(r - g)$

The present value of the growing perpetuity is then the upfront cost: $P = \dfrac{C}{r-g}$.

$$PV(Growing\ Perpetuity) = \frac{C}{r - g}$$

Growing Annuity formula

A growing annuity is a stream of N growing cash flows paid at regular intervals.

The mathematical derivation of the PV formula

The present value of an N-period annuity A with payment C and interest r is given by:

$$PV(GA) = \frac{C}{(1+r)^1} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots + \frac{C(1+g)^{N-1}}{(1+r)^N}$$

$$PV(GA) = \frac{C}{(1+r)} + \frac{C}{(1+r)} * \sum_{n=1}^{N} \frac{(1+g)^n}{(1+r)^n}$$

You may recognize this, from Calculus classes, as a **finite geometric series**. The formula for the sum of such a series is:

$$X = \sum_{n=1}^{N} Z^t = \frac{Z * (1 - Z^N)}{(1 - z)}$$

The Present Value of the N-period annuity can then be written as a geometric progression,

where $Z = \frac{1+g}{1+r}$ and g < r

$$PV(GA) = \frac{C}{(1+r)} + \frac{C}{(1+r)} * \sum_{n=1}^{N} \frac{(1+g)^n}{(1+r)^n} = \frac{C}{(1+r)} + \frac{C}{(1+r)} * \frac{\frac{1+g}{1+r} * \left(1 - \left(\frac{1+g}{1+r}\right)^N\right)}{\left(1 - \frac{1+g}{1+r}\right)}$$

This equation can be simplified by multiplying it by $\frac{1+r}{1+r}$ which is to multiply it by 1. The formula is now reduced to:

$$PV(Growing\ Annuity) = \frac{C}{r-g} * \left[1 - \left(\frac{1+g}{1+r}\right)^N\right] \ ; \ \ with\ g < r$$

The formula for the growing annuity encompasses all of the other formulas

$$PV(Growing\ Annuity) = \frac{C}{r - g} * \left(1 - \left(\frac{1 + g}{1 + r}\right)^N\right)$$

Example:

Consider a growing perpetuity:

⇨ $N = \infty\ ; g < r$

⇨ $\frac{1+g}{1+r} < 1$

⇨ $\left(\frac{1+g}{1+r}\right)^N \to 0\ as\ N \to \infty$

⇨ $PV(Growing\ Perpetuity) = \frac{C}{r-g} * (1 - 0) = \frac{C}{r-g}$

Type of cash-flows	Constant cash-flows	Growing cash-flows
Perpetuities (last forever)	$g = 0; n \to \infty$ $PV(Perpetuity) = \dfrac{C}{r}$	$g < r; n \to \infty$ $PV(Growing\ Perpetuity) = \dfrac{C}{r - g}$
Annuities (N periods)	$g = 0; n \to N$ $PV(Annuity) = \dfrac{C}{r} * \left[1 - \dfrac{1}{(1 + r)^N}\right]$	$g < r; n \to N$ $PV(Growing\ Annuity) = \dfrac{C}{r - g} * \left[1 - \left(\dfrac{1 + g}{1 + r}\right)^N\right]$